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SYMMETRIES AND EXACT SOLUTIONS OF THE SHALLOW WATER EQUATIONS FOR A TWO-DIMENSIONAL SHEAR FLOW

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This paper considers nonlinear equations describing the propagation of long waves in two-dimensional shear flow of a heavy ideal incompressible fluid with a free boundary. A nine-dimensional group of transformations admitted by the equations of motion is found by symmetry methods. Two-dimensional subgroups are used to find simpler integrodifferential submodels which define classes of exact solutions, some of which are integrated. New steady-state and unsteady rotationally symmetric solutions with a nontrivial velocity distribution along the depth are obtained.

Key words: *symmetry, exact solutions, two-dimensional shear flows, long waves.*

Introduction. Approximate models of shallow-water theory are used to model wave processes in fluids and to describe large-scale motions in the atmosphere and ocean. A mathematical foundation for the classical (depth-averaged) shallow-water approximation was given by Ovsyannikov [1]. The long-wave model taking into account velocity shear along the depth, especially in the two-dimensional case has been studied to a lesser extent. The nonlinear equations of rotational shallow water for plane-parallel motions were studied in [2–7], etc., where infinite series of conservation laws were found, classes of exact solutions were constructed, and conditions for the generalized hyperbolicity and well-posedness of the Cauchy problem were formulated. Teshukov [8, 9] studied the shallow-water equations for two-dimensional shear flow, established the existence of simple waves, constructed an extension of Prandtl–Meyer waves, and formulated conditions for the generalized hyperbolicity of the steady-state equations.

In the present work, a theoretical group analysis of the two-dimensional shallow-water equations for shear flows was performed. The 9-dimensional group of the admitted transformations was found. It was established that the Lie algebra of operators L_9 corresponding to these transformations is isomorphic to the Lie algebra of the admitted operators for the equations of two-dimensional isentropic motion of a polytropic gas with an adiabatic exponent $\gamma = 2$, for which the optimal system subalgebras [10] is known. New classes of exact solutions were constructed using an optimal system of subalgebras that allows a classification of submodels. Steady-state rotationally symmetric solutions describing motion with zones of return flow were obtained. Stable unsteady shear solutions describing the spread (collapse) of a parabolic cavity were found.

1. Mathematical Model and Admitted Transformations. The solutions of the system of differential equations

$$u_t + uu_x + vu_y + wu_z + \rho^{-1}p_x = 0,$$

$$v_t + uv_x + vv_y + wv_z + \rho^{-1}p_y = 0, \qquad \rho^{-1}p_z = -g, \tag{1}$$

$$u_x + v_y + w_z = 0 \qquad [-\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 \le z \le h(t,x,y)]$$

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with the boundary conditions

$$z = 0\colon \quad w = 0, \qquad z = h(t,x,y)\colon \quad h_t + uh_x + vh_y = w, \quad p = p_0 \tag{2}$$

describe the unsteady three-dimensional motion of an ideal incompressible fluid with a free boundary above an even bottom in a gravity field in the long-wave approximation. Model (1) follows from the exact Euler equations in the long-wave limit $\varepsilon = H_0/L_0 \to 0$, where H_0 and L_0 are the characteristic vertical scale and the characteristic wave length. The dimensionless variables t, x, y, z, u, v, w, p, and h correspond to time, Cartesian coordinates, velocity components, pressure, and fluid layer depth; the dimensionless constants ρ and g are the fluid density and the acceleration due to gravity (without loss of generality, one can set $g = 1$). By virtue of the third equation of system (1), the fluid pressure is hydrostatic and is distributed along the depth as

$$p = \rho g(h - z) + p_0$$

(taking into account the dynamic condition on the free boundary), which allows the pressure to be eliminated from Eqs. (1).

To study the symmetry properties of the examined model, it is reasonable to simplify the kinematic condition on the free boundary using the change of variables

$$z' = \frac{z}{h(t,x,y)}, \qquad w' = \frac{dz'}{dt} = \frac{1}{h}\Big(w - \frac{z}{h}\,(h_t + uh_x + vh_y)\Big).$$

In the new variables, system (1) becomes

$$\begin{gathered} u_t + uu_x + vu_y + w'u_{z'} + gh_x = 0, \qquad v_t + uv_x + vv_y + w'v_{z'} + gh_y = 0, \\ h_t + uh_x + vh_y + h(u_x + v_y + w'_{z'}) = 0, \qquad h_{z'} = 0. \end{gathered} \tag{3}$$

The boundary conditions (2) are written as

$$w'\Big|_{z'=0} = 0, \qquad w'\Big|_{z'=1} = 0. \tag{4}$$

System (3) includes the equation $h_{z'} = 0$. which implies that all unknown functions $\boldsymbol{u}$ depend on all independent variables $\boldsymbol{x}$.

According to the general theory of group analysis [11], we define the infinitesimal operator X and its first continuation Y:

$$X = \xi^i(\boldsymbol{x},\boldsymbol{u})\,\partial_{x^i} + \eta^j(\boldsymbol{x},\boldsymbol{u})\,\partial_{u^j}, \qquad Y = X + \zeta_i^j\,\partial_{u_i^j} \qquad (i,j = 1,\dots,4).$$

Here

$$\boldsymbol{x} = (x^1,\dots,x^4) = (t,x,y,z'), \qquad \boldsymbol{u} = (u^1,\dots,u^4) = (u,v,w',h),$$

$$\zeta_i^j = D_i\eta^j - u_i^j D_i\xi^j, \qquad D_i = \frac{\partial}{\partial x^i} + u_i^j\,\frac{\partial}{\partial u^j} \qquad \Big(u_i^j = \frac{\partial u^j}{\partial x^i}\Big),$$

the summation is performed over repeated indices. To calculate the group of transformations admitted by system (3), we subject it to the first continuation of the operator X and pass to the set of solutions of system (3). As a result, we obtain a system of determining equations for the required functions $\xi^i(\boldsymbol{x},\boldsymbol{u})$ and $\eta^j(\boldsymbol{x},\boldsymbol{u})$ that admits splitting in the variables u_i^j. Omitting the bulky intermediate calculations, result of calculation of the group is as follows:

$$\begin{gathered} \xi^1 = a_1t^2 + a_2t + a_3, \qquad \xi^2 = a_1tx + b_1x + b_2y + b_3t + b_4, \\ \xi^3 = a_1ty - b_2x + b_1y + c_1t + c_2, \qquad \xi^4 = d_1z' + d_2(t,x,y), \\ \eta^1 = (b_1 - a_2 - a_1t)u + b_2v + a_1x + b_3, \\ \eta^2 = -b_2u + (b_1 - a_2 - a_1t)v + a_1y + c_1, \\ \eta^3 = d_{2x}u + d_{2y}v + d_{2t} + (d_1 - a_2 - a_1t)w', \qquad \eta^4 = 2(b_1 - a_2 - a_1t)h. \end{gathered} \tag{5}$$

TABLE 1

Commutators L_9

Operator	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
X_1	0	0	0	0	X_2	X_1	0	X_3	X_1
X_2	0	0	0	0	$-X_1$	X_2	0	X_4	X_2
X_3	0	0	0	0	X_4	X_3	$-X_1$	0	$-X_3$
X_4	0	0	0	0	$-X_3$	X_4	$-X_2$	0	$-X_4$
X_5	$-X_2$	X_1	$-X_4$	X_3	0	0	0	0	0
X_6	$-X_1$	$-X_2$	$-X_3$	$-X_4$	0	0	0	0	0
X_7	0	0	X_1	X_2	0	0	0	X_9	$2X_7$
X_8	$-X_3$	$-X_4$	0	0	0	0	$-X_9$	0	$-2X_8$
X_9	$-X_1$	$-X_2$	X_3	X_4	0	0	$-2X_7$	$2X_8$	0

Here a_i, b_i, c_1, and d_1 are constants. We require that boundary conditions (4) be invariant under the obtained transformations (5) admitted by Eqs. (3). It is easy to see that this requirement leads to the following constraints: $d_1 = 0$ and $d_2 = 0$. Thus, system (3), (4) admits the Lie algebra of the operators L_9:

$$X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = t\,\partial_x + \partial_u,$$

$$X_4 = t\,\partial_y + \partial_v, \qquad X_5 = -y\,\partial_x + x\,\partial_y - v\,\partial_u + u\,\partial_v,$$

$$X_6 = x\,\partial_x + y\,\partial_y + u\,\partial_u + v\,\partial_v + 2h\,\partial_h, \qquad X_7 = \partial_t, \tag{6}$$

$$X_8 = t^2\,\partial_t + tx\,\partial_x + ty\,\partial_y + (x - tu)\,\partial_u + (y - tv)\,\partial_v - 2tw'\,\partial_{w'} - 2th\,\partial_h,$$

$$X_9 = 2t\,\partial_t + x\,\partial_x + y\,\partial_y - u\,\partial_u - v\,\partial_v - 2w'\,\partial_{w'} - 2h\,\partial_h.$$

It was established that the shallow-water equations for two-dimensional shear flow admit translations in time and the horizontal two-dimensional variables, Galilean translations in x and y, two stretchings, rotation about the z axis, and nontrivial projective transformation.

In the theoretical group analysis of the model, an important and labor-consuming step is to construct an optimal system of subalgebras [12] for the obtained Lie algebra of operators. This can be done invoking the results of the studies performed using the SUBMODEL program [13] at the Institute of Hydrodynamics of the Siberian Division of the Russian Academy of Sciences. From Table 1, it follows that the Lie algebra L_9 of operators (6) is decomposed into the semidirect sum of the radical $J = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ and the Levy factor $N = \{X_7, X_8, X_9\}$. The Lie algebra L_9 coincides with the Lie algebra of the operators admitted by the equations of two-dimensional isentropic motion of a polytropic gas with $\gamma = 2$, for which an optimal system of subalgebras is constructed in [10]. The system is optimal the sense that the solutions obtained by means of its representatives exhaust all possible invariant and partially invariant solutions to within the change of variables.

To construct and analyze the exact solutions of the long-wave model for two-dimensional shear flow using the obtained symmetries (6), it is convenient to pass to semi-Lagrangian coordinates (x, y, λ). This passage is performed by changing the vertical Eulerian variable $z = \Phi(t, x, y, \lambda)$, where the function $\Phi(t, x, y, \lambda)$ is a solution of the Cauchy problem [3]

$$\Phi_t + u(t, x, y, \Phi)\Phi_x + v(t, x, y, \Phi)\Phi_y = w(t, x, y, \Phi),$$

$$\Phi(0, x, y, \lambda) = \lambda h(0, x, y) \qquad (0 \le \lambda \le 1).$$

Then, we obtain the following integrodifferential system of equations for the required quantities $u(t, x, y, \lambda)$, $v(t, x, y, \lambda)$, and $H(t, x, y, \lambda) = \Phi_\lambda$ [8]

$$u_t + uu_x + vu_y + g\int_0^1 H_x\,d\lambda = 0, \quad v_t + uv_x + vv_y + g\int_0^1 H_y\,d\lambda = 0, \quad H_t + (uH)_x + (vH)_y = 0. \tag{7}$$

The change of variables is reversible if the condition $\Phi_\lambda > 0$ is satisfied.

We denote by X_i' the operators admitted by system (7) that correspond to the transformations X_i specified in Eulerian coordinates in (6). The operators X_i' have the form

$$X_i' = X_i \quad (i = 1, \ldots, 5, 7), \qquad X_6' = x\,\partial_x + y\,\partial_y + u\,\partial_u + v\,\partial_v + 2H\,\partial_H,$$

$$X_8' = t^2\,\partial_t + tx\,\partial_x + ty\,\partial_y + (x - tu)\,\partial_u + (y - tv)\,\partial_v - 2tH\,\partial_H,$$

$$X_9' = 2t\,\partial_t + x\,\partial_x + y\,\partial_y - u\,\partial_u - v\,\partial_v - 2H\,\partial_H.$$

The obtained symmetries of the shallow-water equations for two-dimensional shear flows and the optimal system of subalgebras constructed in [10] and containing 179 representatives allow the construction of invariant and partially invariant solutions of the model. In the case of no velocity shear along the depth ($u_\lambda = v_\lambda = 0$ and $H = h$), model (7) reduces to the two-dimensional equations of isentropic gas dynamics for a polytropic gas with an adiabatic exponent $\gamma = 2$ and all solutions of this model of gas dynamics are a particular class of solutions of the more general system (7). Thus, of greatest interest is to obtain solutions with nontrivial velocity distributions along the depth that take into account the specificity of shear flows.

2. Rotationally Symmetric Submodels. In the paper, we consider all submodels constructed from two-dimensional representatives of the optimal system of subalgebras θL_9 and containing the rotation operator as one of the basic operators. There are five such representatives (the optimal system includes a total of 34 two-dimensional subalgebras). Below, we give invariant rotationally symmetric submodels of Eqs. (7) and representations of solutions using the following notation for polar coordinates and the radial and circumferential velocity components:

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{y}{x}, \qquad U = \frac{xu + yv}{\sqrt{x^2 + y^2}}, \qquad V = \frac{xv - yu}{\sqrt{x^2 + y^2}}.$$

1. The submodel constructed from the subalgebra (X_5, X_6) is given by

$$\varphi_t + \varphi^2 - \psi^2 + 2g\int_0^1 \eta\,d\lambda = 0, \qquad \psi_t + 2\varphi\psi = 0, \qquad \eta_t + 4\varphi\eta = 0. \tag{8}$$

The solution is represented as

$$U = r\varphi(t, \lambda), \qquad V = r\psi(t, \lambda), \qquad H = r^2\eta(t, \lambda).$$

2. The submodel constructed from the subalgebras (X_5, X_7) is given by

$$\frac{\partial}{\partial r}\Big(\frac{U^2}{2} + g\int_0^1 H\,d\lambda\Big) = \frac{V^2}{r}, \qquad U\,\frac{\partial\,(rV)}{\partial r} = 0, \qquad \frac{\partial\,(rUH)}{\partial r} = 0 \tag{9}$$

(U, V, and H are functions of the variables r and λ).

3. The submodel constructed from the subalgebras $(X_5, X_6 + X_7)$ is given by

$$\zeta(\varphi - 1)\varphi_\zeta + \varphi^2 - \psi^2 + g\int_0^1 (2\eta + \zeta\eta_\zeta)\,d\lambda = 0,$$

$$\zeta(\varphi - 1)\psi_\zeta + 2\varphi\psi = 0, \qquad \zeta(\varphi - 1)\eta_\zeta + \zeta\eta\varphi_\zeta + 4\varphi\eta = 0.$$

The solution is represented as

$$U = r\varphi(\zeta, \lambda), \qquad V = r\psi(\zeta, \lambda), \qquad H = r^2\eta(\zeta, \lambda), \qquad \zeta = r\exp(-t).$$

4. The submodel constructed from the subalgebras $(X_5, bX_6 + X_7 + X_8)$ is given by

$$(\varphi - b\zeta)\zeta\varphi_\zeta + b\zeta\varphi - \psi^2 + \zeta^2 + g\zeta\int_0^1 \eta_\zeta\,d\lambda = 0,$$

$$(\varphi - b\zeta)\zeta\psi_\zeta + (\varphi + b\zeta)\psi = 0, \qquad (\varphi - b\zeta)\zeta\eta_\zeta + \eta\varphi_\zeta + (\varphi + 2b\zeta)\eta = 0.$$

The solution is represented as

$$U = \frac{\varphi(\zeta,\lambda)}{\sqrt{t^2+1}} \exp(b \arctan t) + \frac{rt}{t^2+1}, \qquad V = \frac{\varphi(\zeta,\lambda)}{\sqrt{t^2+1}} \exp(b \arctan t),$$

$$H = \frac{\eta(\zeta,\lambda)}{t^2+1} \exp(2b \arctan t), \qquad \zeta = \frac{r}{\sqrt{t^2+1}} \exp(-b \arctan t) \qquad (b \ge 0).$$

5. The submodel constructed from the subalgebra $\langle X_5, bX_6 + X_9 \rangle$ is given by

$$(2\varphi - (b+1)\zeta)\zeta\varphi_\zeta + (b-1)\zeta\varphi - 2\psi^2 + 2g\zeta \int_0^1 \eta_\zeta \, d\lambda = 0,$$

$$(2\varphi - (b+1)\zeta)\zeta\psi_\zeta + (2\varphi + (b-1)\zeta)\psi = 0, \qquad (2\varphi - (b+1)\zeta)\zeta\eta_\zeta + 2\zeta\eta\varphi_\zeta + 2(b-1)\zeta\eta + 2\varphi\eta = 0.$$

The solution is represented as

$$U = \varphi(\zeta,\lambda)t^{(b-1)/2}, \qquad V = \psi(\zeta,\lambda)t^{(b-1)/2},$$

$$H = \eta(\zeta,\lambda)t^{b-1}, \qquad \zeta = rt^{(b+1)/2} \qquad (b \ge 0).$$

3. Spread (Collapse) a Parabolic Cavity. For the integrodifferential submodel (8), a class of solutions is obtained that describes the shear motion of a fluid for broadening or narrowing of a parabolic cavity. The occurrence of the spread or collapse modes is determined by the initial distribution of the velocity field. It was found that the obtained solution does not contain complex characteristic roots, which is a necessary condition for flow stability.

Introducing the function

$$w(t,\lambda) = \exp\Big(\int_0^t \varphi(t',\lambda)\, dt'\Big),$$

we reduce system (8) to one second-order integrodifferential equation

$$w_{tt} - \psi_0^2(\lambda)w^{-3} + 2gw \int_0^1 \eta_0(\lambda) w^{-4}\, d\lambda = 0. \tag{10}$$

Thus functions φ, ψ, and η are expressed in terms of w as follows:

$$\varphi = w^{-1}w_t, \qquad \psi = \psi_0(\lambda)w^{-2}, \qquad \eta = \eta_0(\lambda)w^{-4}$$

[$\psi_0(\lambda)$ and $\eta_0(\lambda)$ are arbitrary functions]. The structure of Eq. (10) for $\psi_0(\lambda) = 0$ allows the solution to be sought in the form $w(t,\lambda) = \sum_{i=1}^n a_i(\lambda)b_i(t)$ with arbitrary functions $a_i(\lambda)$ and $\eta_0(\lambda)$. Then, we obtain the following system of second-order ordinary differential equations for the functions $b_i(t)$:

$$b_i'' + 2gb_i \int_0^1 \eta_0(\lambda)\Big(\sum_{i=1}^n a_i(\lambda)b_i(t)\Big)^{-4} d\lambda = 0 \qquad (i = 1,\ldots,n).$$

Below, we consider the case $n = 2$ and give an analytical solution in parametric form (for $n = 1$, integration of the equation leads to a solution without velocity shear along the depth). The functions $a_1(\lambda)$ and $a_2(\lambda)$ are specified so as to satisfy the inequalities $l(\lambda) = a_2(\lambda)a_1^{-1}(\lambda) > 0$ and $l'(\lambda) > 0$; in this case, $\eta_0(\lambda) = a_1^4(\lambda)l'(\lambda)/(2g)$. Then, we obtain the following system for the functions $b_1(t)$ and $b_2(t)$:

$$\begin{aligned} b_1'' + \frac{b_1}{3b_2}\Big(\frac{1}{(b_1 + l_0b_2)^3} - \frac{1}{(b_1 + l_1b_2)^3}\Big) &= 0, \\ b_2'' + \frac{1}{3}\Big(\frac{1}{(b_1 + l_0b_2)^3} - \frac{1}{(b_1 + l_1b_2)^3}\Big) &= 0. \end{aligned} \tag{11}$$

Here $l_0 = l(0)$ and $l_1 = l(1)$. From the relation $b_2 b_1'' - b_1 b_2'' = 0$, which is a consequence of Eqs. (11), we find the integral $b_2 b_1' - b_1 b_2' = k_1 = \text{const}$. By means of the change of variables

$$m(\tau) = \frac{b_1(t(\tau))}{b_2(t(\tau))}, \qquad F(\tau) = \frac{1}{b_2(t(\tau))}, \qquad t'(\tau) = \frac{1}{F^2(\tau)}$$

and with the use of the above integral, Eqs. (11) become

$$m'(\tau) = k_1, \qquad F''(\tau) = \frac{F(\tau)}{3}\Big(\frac{1}{(m(\tau)+l_0)^3} - \frac{1}{(m(\tau)+l_1)^3}\Big).$$

The first equation is easily integrated: $m(\tau) = k_1\tau + k_2$, and the second reduces to the Riccati equation

$$s' + s^2 = \frac{1}{3}\Big(\frac{1}{(k_1\tau + k_2 + l_0)^3} - \frac{1}{(k_1\tau + k_2 + l_1)^3}\Big), \tag{12}$$

where $s(\tau) = F'(\tau)/F(\tau)$. To obtain the solution of Eqs. (11) in parametric form

$$b_1(t) = \frac{k_1\tau + k_2}{F(\tau)}, \qquad b_2(t) = \frac{1}{F(\tau)}, \qquad F(\tau) = \exp\Big(s(0) + \int_0^\tau s(\tau')\,d\tau'\Big), \qquad t = \int_0^\tau \frac{d\tau'}{F^2(\tau')}$$

(k_i are arbitrary constants), it is necessary to integrate the first-order ordinary differential equation of the (12). It is difficult to obtain a solution of Eq. (12) in explicit form, but it is easy to perform its qualitative analysis and numerical integration. Thus, the constructed class of solutions is given by the formulas

$$U(t,r,\lambda) = \frac{b_1'(t) + l(\lambda)b_2'(t)}{b_1(t) + l(\lambda)b_2(t)}\,r, \qquad V(t,r,\lambda) = 0,$$

$$H(t,r,\lambda) = \frac{l'(\lambda)r^2}{2g(b_1(t) + l(\lambda)b_2(t))^4}, \tag{13}$$

where $l(\lambda)$ is an arbitrary function. The free-boundary equation

$$z = h(t,r) = \frac{r^2}{6gb_2(t)}\Big(\frac{1}{(b_1(t) + l_0 b_2(t))^3} - \frac{1}{(b_1(t) + l_1 b_2(t))^3}\Big)$$

is the equation of an elliptic paraboloid (at each fixed time). The condition $H = \Phi_\lambda > 0$, which guarantees reversibility of the change $z = \Phi(t,x,y,\lambda)$, is satisfied. We note that solution (13) is a shear one ($U_\lambda \neq 0$) if the arbitrary constant k_1 is different from zero.

Let us determine what modes of motion are described by the solution obtained. Let $k_1 > 0$ and $k_2 + l_i > 0$ ($i = 0, 1$). Then, the spread of the parabolic cavity and retardation of the flow occurs in infinite time. The depth of the fluid layer h and the radial velocity component U at each fixed point tend to zero as $t \to \infty$. Figures 1 and 2 show the distributions of the fluid layer depth versus radius and time, and Fig. 3 shows the distribution of the radial velocity versus depth. Figures 1–3 were obtained for the following parameters:

$$k_1 = 1, \qquad k_2 = 0, \qquad l(\lambda) = (\lambda + 1)/2. \tag{14}$$

For the given function $l(\lambda)$, the relation between the Lagrangian variable λ and the vertical Eulerian z variable is written as

$$\lambda = \frac{1}{b_2}\Big(\frac{1}{(b_1 + b_2/2)^3} - \frac{6gb_2}{r^2}z\Big)^{-1/3} - \frac{b_1 + b_2/2}{b_2}.$$

If $k_1 < 0$ and $k_2 + l_i > 0$ ($i = 0, 1$), collapse of the initial parabolic cavity occurs (on each circle $r = \text{const} > 0$, the fluid layer depth increases without bound with time). The parabolic cavity collapses in finite time $t_* = t(\tau_*)$ [the value $\tau_* = -(k_2 + l_0)/k_1$ corresponds to the infinite right side of Eq. (12)]. If $k_2 + l_1 > 0$ and $k_2 + l_0 < 0$, collapse occurs for any sign of the constant k_1.

We analyze the stability of the constructed class of unsteady rotationally symmetric solutions. The necessary and sufficient conditions for the hyperbolicity of the integrodifferential system of equations describing plane–parallel shear motion of an ideal fluid are formulated in [6]. The results obtained in [6] are easily extended to the rotationally symmetric motion described by the equations

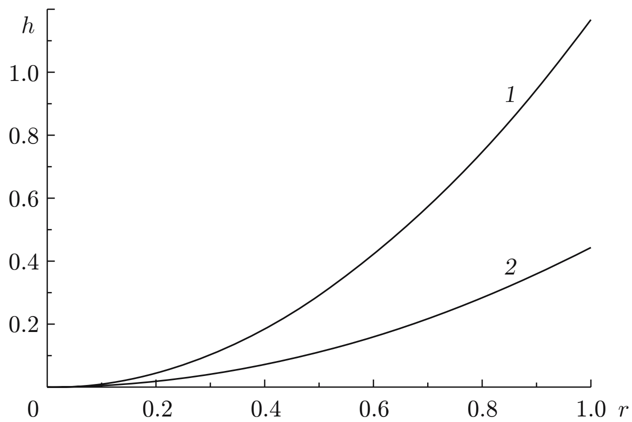

Fig. 1

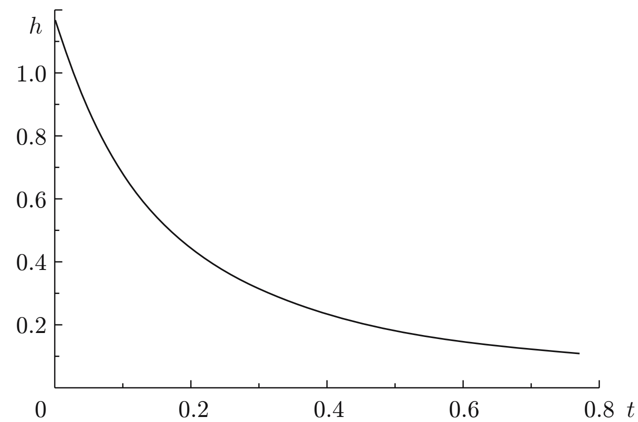

Fig. 2

Fig. 1. Fluid layer depth h versus radius r for $t = 0$ (1) and 0.2 (2).

Fig. 2. Fluid layer depth h versus time t at $r = 1$.

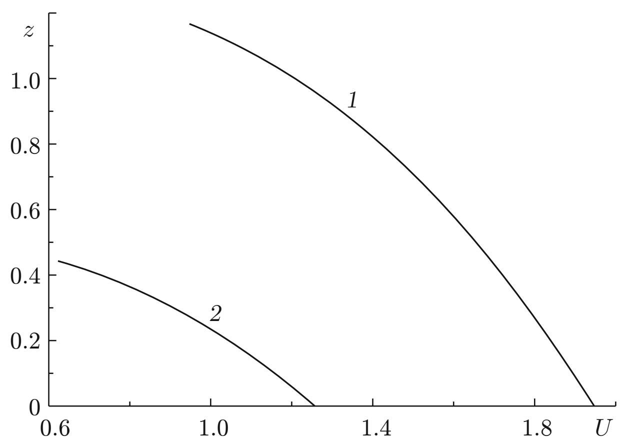

Fig. 3. Distribution of the radial velocity U along the depth at $r = 1$ and $t = 0$ (1) and 0.2 (2).

$$U_t + UU_r + g\int_0^1 H_r\,d\lambda = \frac{V^2}{r}, \qquad V_t + UV_r = -\frac{UV}{r}, \qquad H_t + (UH)_r = -\frac{UH}{r}. \tag{15}$$

In the case of motion with a velocity profile ($U_\lambda \neq 0$) monotonic along the depth, Eqs. (15) are generalized hyperbolic equations if the following conditions are satisfied:

$$\Delta \arg(\chi^+/\chi^-) = 0, \qquad \chi^+ \neq 0. \tag{16}$$

Here

$$\chi^{\pm}(U(\lambda)) = 1 + \frac{g}{\Omega_1}\,\frac{1}{U_1 - U(\lambda)} - \frac{g}{\Omega_0}\,\frac{1}{U_0 - U(\lambda)} - g\int_0^1 \Big(\frac{1}{\Omega(\nu)}\Big)_\nu \frac{d\nu}{U(\nu) - U(\lambda)} \mp \frac{\pi i}{U_\lambda(\lambda)}\Big(\frac{1}{\Omega(\lambda)}\Big)_\lambda,$$

the increment of the argument is calculated for λ ranging from 0 to 1 and fixed values of t and r; $\Omega = U_\lambda/H$; the subscripts 0 and 1 denote the values of the corresponding functions for $\lambda = 0$ and $\lambda = 1$. Conditions (16) guarantee the absence of complex roots of the equation

$$\chi(c) = 1 - g\int_0^1 \frac{H\,d\lambda}{(U - c)^2} = 0,$$

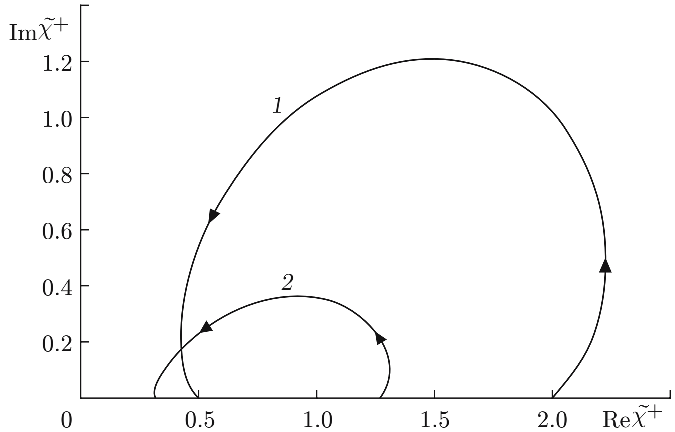

Fig. 4. Real and imaginary parts of the function $\tilde{\chi}^+$ for λ ranging from 0 to 1; 1) $t = 0$; 2) $t = 0.2$.

which defines the propagation velocities of the characteristics $c(t, r)$ (there are a continuous spectrum of characteristic velocities $c = c^\lambda = U$ and a discrete spectrum $c = c^1 < \min\limits_\lambda U$, $c = c^2 > \max\limits_\lambda U$.

To verify conditions (16), we use the functions $\tilde{\chi}^+ = (\varphi_1 - \varphi)(\varphi - \varphi_0)\chi^+$, which, in contrast to $\chi^\pm$, have no poles at the points $\lambda = 0$, 1. We note that the functions $\chi^\pm$ and $\tilde{\chi}^\pm$ do not depend on the variable r; therefore, the conclusion on flow stability is valid for any finite interval of r. Figure 4 shows the real and imaginary parts of the function $\tilde{\chi}^+(t, \lambda)$ at various times for λ ranging from 0 to 1. A plot of the function $\tilde{\chi}^-$ is obtained by reflection about the abscissa. From Fig. 4 it follows that the argument of the complex function $\tilde{\chi}^+$ does not acquire an increment with a change of λ (there is no circulation about zero [14]). This analysis was performed using parameters (14). From calculations performed for solution (13) with different integration constants k_i and the function $l(\lambda)$, it follows that conditions (16) are satisfied for both the spread and collapse of the parabolic cavity. Thus, the solution considered does not contain complex characteristic roots, which is a necessary condition for flow stability.

4. Steady-State Rotationally Symmetric Solutions. The approach developed in studies [5] of plane–parallel flows with a critical layer was used to obtain a class of exact solutions describing two-dimensional flows with return zones.

Integration of submodel (9) leads to the solution

$$U(r, \lambda) = \pm\sqrt{2C_1(\lambda) - r^{-2}C_2^2(\lambda) - 2gh(r)}, \qquad V(r, \lambda) = r^{-1}C_2(\lambda),$$
$$H(r, \lambda) = \frac{C_3(\lambda)}{\sqrt{2r^2C_1(\lambda) - C_2^2(\lambda) - 2gr^2h(r)}}, \tag{17}$$

where the function $h(r)$ is found from the closing relation

$$F(h, r) = h - \int\limits_0^1 \frac{C_3(\lambda)\, d\lambda}{\sqrt{2r^2C_1(\lambda) - C_2^2(\lambda) - 2gr^2h(r)}} = 0.$$

The other solution of Eqs. (9)

$$U = 0, \qquad V = V(r), \qquad h(r) = g^{-1}\int\limits_0^r r'^{-1}V^2(r')\, dr'$$

describes flows without velocity shear along the depth and is not considered further. In (17), the sign changes if the radicand vanishes. We consider the flow region in which $U(r, \lambda) > 0$. Solution (17) includes three arbitrary functions $C_i(\lambda)$. The equality $C_3(\lambda) = 1$ can be achieved by an appropriate choice of the Lagrangian coordinate λ. To simplify the further analysis, we express the function $C_2(\lambda)$ in terms of $C_1(\lambda)$ using the formula $C_2(\lambda) = \sqrt{C_1(\lambda) - C_1(0)}$, and specify the function $C_1(\lambda)$ so as to satisfy the conditions

$$C_1'(\lambda) > 0, \qquad 0 < C_1(0) < 3(g/2)^{2/3}. \tag{18}$$

In this case, the inequality

$$2(C_1(\lambda) - gh)r^2 - C_1(\lambda) + C_1(0) > 0$$

[the radicand in (17) in nonnegative] is satisfied for all $\lambda \in [0, 1]$ if

$$r > 1/\sqrt{2}, \qquad 0 < h < a \qquad [a = C_1(0)/g].$$

We consider the function $F(h, r)$ at the sections $h = h_0$, $h_0 \in (0, a)$. As $r \to \infty$, the function $F(h_0, r)$ tends to the value $h_0 > 0$, and as $r \to 1/\sqrt{2}$, it tends to the value $h_0 - (C_1(0) - gh_0)^{-1/2}$, which is negative by virtue of the condition (18). In addition, for all $r \in (1/\sqrt{2}, \infty)$, the derivative $F_r(h_0, r) > 0$. Therefore, at each section $h = h_0$, the equation $F(h_0, r) = 0$ has a unique root.

We examine the function $F(h, r)$ at the sections $r = r_0$, $r_0 \in (1/\sqrt{2}, \infty)$. As $h \to 0$, the function $F(h, r_0)$ tends to a negative quantity, and as $h \to a$, it tends to the quantity

$$a - \frac{1}{\sqrt{2r_0^2 - 1}} \int_0^1 \frac{d\lambda}{\sqrt{C_1(\lambda) - C_1(0)}},$$

which is positive for large values of r_0 [the integral $\int_0^1 (C_1(\lambda) - C_1(0))^{-1/2}\, d\lambda$ converges]. Because the function $F(h, r_0)$ is convex upward ($F_{hh} < 0$) and changes sign in the interval $h \in [0, a]$, it follows that for large values of r_0, the equation $F(h, r_0) = 0$ has one root.

We find the root $r = r_*$ of the equations $F(a, r) = 0$. It is easy to see that

$$r_* = \sqrt{\frac{b^2 + 1}{2}} > \frac{1}{\sqrt{2}} \qquad \Big(b = a \int_0^1 \frac{d\,\lambda}{\sqrt{C_1(\lambda) - C_1(0)}}\Big).$$

Because $F_{hh}(h, r_*) < 0$, $F(0, r_*) < 0$, $F(a, r_*) = 0$, and $F_h(h, r_*) \to -\infty$ as $h \to a$, in the interval $(0, a)$, the equation $F(h, r_*) = 0$ has one more root. Thus, an interval $r_0 \in (d, r_*)$ exists in which the equation $F(h, r_0)$ has two roots at the sections $r = r_0$ [$d \geq 1/\sqrt{2}$ is the minimum value of r for at which the equation $F(h, d) = 0$ has a root].

The derivative of the function $h = h(r)$, which is given implicitly by the equation $F(h, r) = 0$, is calculated by the formula $h'(r) = F_r/F_h$ and becomes infinite at the point $r = d$, where $F_h = 0$. According to the definition of the characteristics for systems of equations with operator coefficients [6], the surface $r = d$ is a characteristic if the examined solution satisfies the equality

$$F_h = 1 - gr^2 \int_0^1 [2r^2(C_1(\lambda) - gh) + C_1(0) - C_1(\lambda)]^{-3/2}\, d\lambda \Big|_{r=d} = 0.$$

Thus, solution (17) is determined for $r > d$ and is bounded by the characteristic $r = d$.

The solution of the equation $F(h, r) = 0$ corresponding to the functions

$$C_1(\lambda) = g(\lambda + 1), \qquad C_2(\lambda) = \sqrt{g\lambda}, \qquad C_3(\lambda) = 1, \qquad g = 1, \tag{19}$$

is shown in Fig. 5. With a different choice of the function $C_1(\lambda)$ subject to conditions (18), the plot does not change qualitatively.

From the aforesaid it follows that, for $r > d$, the equation $F(h, r) = 0$ has two branches of solutions. The lower branch $h = h_1(r)$ is defined for all $r > d$; the upper branch $h_2(r)$ is defined for $r \in [d, r_*]$. We continue the solution (17) with the function $h = h_2(r)$ to the region $r > r_*$. As a result, we have a steady-state solution that describes flow with the critical layer [on a certain line in the flow region, the velocity $U(r, \lambda)$ vanishes]. For $r > r_*$, we specify the free-boundary equation $z = h_2(r)$ arbitrarily, requiring that the following conditions be satisfied:

$$h_2(r_*) = a, \qquad h_2'(r_*) = 0, \qquad h_2(r) \geq h_1(r). \tag{20}$$

If, for a certain $r = r^* > r_*$, the equality $h_2(r^*) = h_1(r^*)$ is satisfied, we additionally require that the equality $h_2'(r^*) = h_1'(r^*)$ be satisfied. In this case, for $r > r^*$, solution (17) with the functions $C_i(\lambda)$ and $h = h_1(r)$ chosen above takes place.

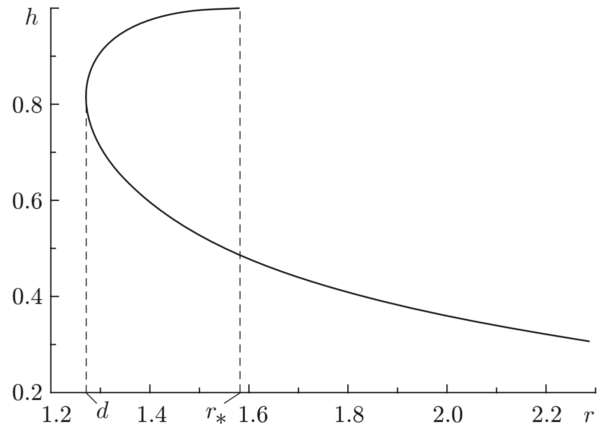

Fig. 5. Typical plot of the solution of the equation $F(h, r) = 0$.

For $r \geq r_*$, using the equation $z = f(r)$, where

$$f(r) = h_2(r) - \int_0^1 \frac{d\lambda}{\sqrt{2r^2(C_1(\lambda) - gh_2(r)) + C_1(0) - C_1(\lambda)}}, \tag{21}$$

we specify the upper boundary of the region of return flow; the lower boundary of the region is given by $z = 0$. In the region $0 \leq z \leq f(r)$, we construct a flow that possesses the following property: on a certain curve in this region, the function U changes sign. For $r > r_*$, the solution in the external region (from the boundary of the region of return flow to the free boundary) is defined by formulas (17) with the plus sign before the square root and the function $h = h_2(r)$, and in the region of return flow, it is defined by the formulas

$$U(r, \lambda) = \mp\sqrt{2(Q(\lambda) - gh_2(r))}, \qquad V(r, \lambda) = 0, \qquad H(r, \lambda) = -(rU)^{-1}. \tag{22}$$

Here $Q(\lambda)$ is an unknown function; the minus sign is taken for $0 \leq \lambda \leq \mu$, and the plus sign for $\mu \geq \lambda \geq 0$. The value of $\mu(r)$ is given by the equation $Q(\mu) - gh_2(r) = 0$.

Integration of the function H on λ from 0 to μ yields the height of the line on which $U = 0$ [this line is given by the equation $z = f(r)/2$]. Further integration from μ to 0 in the region above the line $U = 0$ yields the depth of the region of return flow. Equating this quantity to the known function $f(r)$ given by formula (21), we obtain the integral equation for $Q(\lambda)$:

$$\frac{\sqrt{2}}{r} \int_0^\mu \frac{d\lambda}{\sqrt{Q(\lambda) - gh_2(r)}} = f(r). \tag{23}$$

Making the change of variables $\eta = gh_2(r)$ and $s = Q(\lambda)$ and using the notation

$$G(\eta) = \frac{r(\eta) f(r(\eta))}{\sqrt{2}}, \qquad \tau(s) = -\frac{1}{Q'(\lambda(s))},$$

we reduce Eq. (23) to the Abel equation

$$-\int_{C_1(0)}^{s} \frac{\tau(s)}{\sqrt{s - \eta}} = G(\eta),$$

whose solution has the form

$$\tau(s) = \frac{1}{\pi} \int_{C_1(0)}^{s} \frac{G'(\eta)}{\sqrt{\eta - s}}. \tag{24}$$

The function $Q(\lambda)$ can be found by integrating the equation

$$\tau(Q)\, dQ + d\lambda = 0, \qquad Q(0) = C_1(0).$$

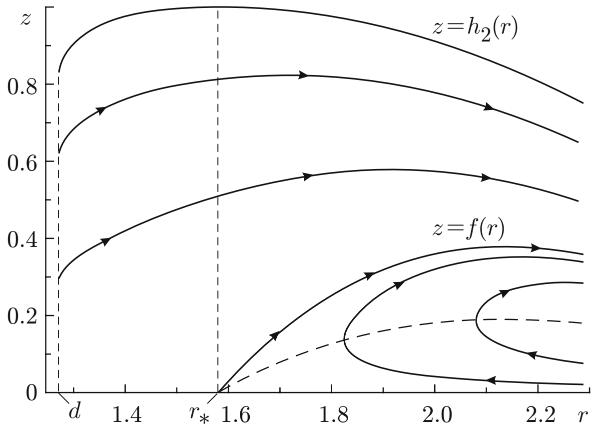

Fig. 6. Streamlines in a steady-state rotationally symmetric solution with a region of return flow (the dashed curve is $U = 0$).

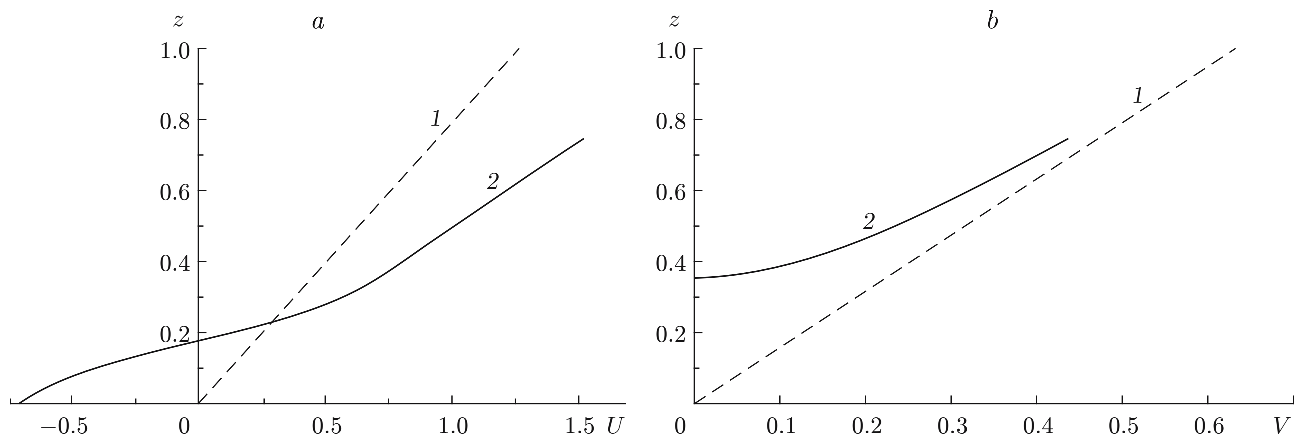

Fig. 7. Distributions of the radial (a) and circumferential (b) velocities along the depth for $r = 1.58$ (1) and 2.29 (2).

Thus, the solution in the region of return flow is constructed and defined by formulas (22). An example of using the above algorithm to construct a solution with a region of return flow is considered below. Let the arbitrary functions included in solution (17) be chosen according to (19). In this case, $d \approx 1.27$ and $r_* = \sqrt{5/2} \approx 1.58$. According to (20), we continue the upper branch of the solution of the equation $F(h, r) = 0$ (see Fig. 5), specifying the function $h = h_2(r)$ in the interval $r \in [r_*, r^*]$:

$$gh_2(r) = 1 - \alpha(r - r_*)^2 \qquad (\alpha > 0, \quad h_2(r^*) > h_1(r^*)).$$

Then, for $r > r_*$, a region of return flow is formed with the upper boundary

$$z = f(r) = \frac{1 - \alpha(r - r_*)^2}{g} - \frac{2}{2r^2 - 1}\Big(\sqrt{2r^2(1 + \alpha(r - r_*)^2) - 1} - \sqrt{\alpha}\, r(r - r_*)\Big).$$

In the region of external flow $f(r) \le z \le h_2(r)$, the solution is defined by formulas (17) with the functions $C_i(\lambda)$ and $h = h_2(r)$ specified above, and in the region of return flow $0 \le z \le f(r)$, it is specified by formulas (22) with the function $Q(\lambda)$ to be determined.

To find $Q(\lambda)$, it is necessary to calculate the singular integral on the right side of (24). A simple analysis shows that the function $G'(\eta)$ is representable as

$$G'(\eta) = W(\eta)(1 - \eta)^{-1/2} \qquad [0 < s \le \eta \le C_1(0) = 1],$$

where $W(\eta)$ is a continuous bounded function. This representation follows from the definition of the function

$$G(\eta) = rf(r)/\sqrt{2}, \qquad r(\eta) = r_* + \sqrt{(1-\eta)/\alpha}.$$

Isolating singularities and changing the variable $\eta = (1-s)\nu + s$, we bring (24) to a form more convenient for numerical integration:

$$\tau(s) = -\frac{1}{\pi}\int\limits_s^1 \frac{W(\eta)\,d\eta}{\sqrt{(\eta - s)(1-\eta)}} = -\frac{1}{\pi}\int\limits_0^1 \frac{\bar{W}(\nu; s)\,d\nu}{\sqrt{\nu(1-\nu)}}.$$

The calculation results presented in Figs. 6 and 7 were obtained for $\alpha = 1/2$ and $r^* = 2.29$. Figure 6 shows the free boundary $z = h_2(r)$, the boundary of the region of return flow $z = f(r)$, the streamlines with an indication of the flow direction (lines $\lambda = \text{const}$), and the line on which $U = 0$. The distributions of the radial and circumferential velocities along the depth at $r = r_*$ (curve 1) and $r = r^*$ (curve 2) are shown in Fig. 7. We note that, for $r > r^*$, the function $h = h_2(r)$ can be continued so as to reach the lower branch $h = h_1(r)$. In this case, the region of return flow exists in an interval which is finite in r.

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